A moving mesh BGK scheme for multi-material flows
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Abstract
In this paper, a moving mesh BGK scheme (MMBGK) for multi-material flow computations is proposed. The basic idea of constructing the MMBGK is to couple the Lagrangian method for the material interface with the remapping-free ALE-type kinetic method within each single material region. Numerical examples in one and two dimensional space are presented to illustrate the efficiency and accuracy of the scheme.

Keywords: Multi-material flows; Moving mesh; BGK scheme; Lagrangian method; Diffusive velocity

1. Introduction
The Eulerian and the Lagrangian methods are two main simulation methods in computational fluid dynamics. The Eulerian method holds fixed mesh and the transports between different volumes rely on the interface fluxes. The Eulerian method is robust in the capturing complicated flow structure with large deformation, but may result in badly smeared material interface and wave structure due to numerical diffusion. For the Lagrangian method, there is no fluids mixing coming from different cells and the material interface can be sharply kept. But, the Lagrangian method may result in mesh distortion and tangling, which causes inaccuracy and even breakdown of computation.

In order to combine the advantages of both Eulerian and Lagrangian methods, the Arbitrary Lagrangian-Eulerian (ALE) technique was developed [2]. Generally speaking, the ALE method falls into the category of moving mesh methods, and the basic idea is that a fixed number of nodes with fixed connectivity is moved within the computational domain to resolve local gradients of the flow variables, and to increase numerical accuracy. Another similar one is the static mesh movement method [8], Harten and Hyman [3] also developed a moving mesh techniques in the past. they began their study in this direction by defining adaptive grid speed in each time step to improve the resolution of shocks and contact discontinuities, this method was then developed by many researchers.

Based on the unified coordinate transformation, a gas-kinetic scheme has been developed [4,5], the unified gas-kinetic method includes the following ingredient, firstly, the gas-kinetic BGK equation is transformed from the physical space to a computational space. More recently, a remapping free gas-kinetic method was proposed for both Euler and Navier-Stokes equations [7], the method incorporates the ALE ingredients into a properly designed mesh velocity to avoid severe mesh distortion.

In this paper, based on the integral form of fluid equations, a moving mesh BGK scheme (MMBGK) for multi-material flow computations is proposed. The methodology here is to combine the Lagrangian method [6] for capturing the material interface with the modified remapping-free ALE method [7] inside a single material region.

2. Numerical scheme for multi-material flows
In this section, based on the integral equations we will construct the corresponding finite volume scheme on a moving reference frame. Lagrangian numerical fluxes across material interfaces and kinetic numerical fluxes on the moving mesh will be given.

The conservative system for the mass, momentum and energy can be written in a unified integral form for the 2D compressible Navier-Stokes equations over a moving control volume with an arbitrary boundary velocity \( U \) in the following form,

\[
\frac{4}{400} \rho dv - \frac{4}{400} \rho (U - U) \cdot nds,
\]

\[
\frac{4}{400} \rho U dv - \frac{4}{400} \rho (U - U) \cdot nds + \frac{4}{400} \nabla \cdot \Pi nds,
\]

\[
\frac{4}{400} \rho E dv - \frac{4}{400} \rho (E - E) \cdot nds + \frac{4}{400} \nabla \cdot \Pi U nds - \frac{4}{400} q \cdot nds,
\]

where \( \rho, U, \rho, \Pi, q \) are the density, velocity, total energy, stress, and heat flux of the fluid respectively. Here \( S(t) \) is the boundary of \( \Omega \), and \( n \) is the outward unit normal vector of \( S(t) \). Note that the above equations become the Eulerian form when \( U_g = 0 \), and the Lagrangian form when \( U = u \). When omitting the viscosity and heat conductivity, the above Navier-Stokes equations reduce to the Euler equations. In general, the boundary velocity \( U \) is a function of time and position, which controls the evolution of the volume \( \Omega \), and the mesh velocity is arbitrary.
Suppose the control volume $\Omega$ is qua drangle domain, let $\Omega = \Omega(t^{n+1})$ and $\Omega = \Omega(t^n)$, integrating the time variable in, we get the discrete form of for a domain with mesh velocity $(U_g, V_g)$:

\[
\begin{align*}
\frac{dF}{dt} = &\sum_{i} F_{\rho\psi_e}(t), \\
\frac{d}{dt} \rho UdV = &\sum_{i} (F_{\rho\psi_e}(t) + U_g F_{\rho\psi_e}(t)), \\
\frac{d}{dt} \rho Edv = &\sum_{i} (F_{\rho\psi_e}(t) + U_g F_{\rho\psi_e}(t) + \frac{1}{2} U^2_g F_{\rho\psi_e}(t)).
\end{align*}
\]

where $F_{\psi_e}$ is the flux of the variables across the edge $e_i$ over the time interval.

We construct the fluxes $F_{\psi_e}$ in the above on a moving mesh by using a kinetic BGK scheme. In [7], we have constructed the fluxes on the moving mesh by the directed discretization of the fluid equations, we can also construct the fluxes on the moving mesh by movement of microscope particles inside a single material region, and the details are omitted.

The BGK model in one dimensions can be written as (see [9])

\[
f_{\psi} + uf_{\psi} = (g - f) / \tau
\]

where $f$ is the gas distribution function and $g$ is the equilibrium state approached by $f$, $(u,v)$ is the particle velocity. The equilibrium state is a Maxwellian distribution

\[
g = \rho \left( \frac{\lambda}{\pi} \right)^{K+1} \sum_{i=0}^{K} e^{-\lambda((u-U)^2 + \xi^2)}
\]

where $(U,V)$ is the macro fluid velocity, $\lambda = \frac{m}{2kT}$ with $m$, $k$ and $T$ being the molecular mass, the Boltzmann constant and the temperature, respectively. The total number of degree of freedom $K$ is equal to $\frac{5}{\gamma} - 3\gamma + 1$ with $\gamma$ being the specific heat ratio, and denotes

\[
\xi^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_K^2.
\]

For multi-material flow simulations, a purely Lagrangian method easily leads to mesh distortion and tangling. Here the Lagrangian method will be only used on material interfaces in order to keep the interface sharp, while the mesh within each single component flow can move freely. The basic idea of Lagrangian method for interface comes from Abgrall and Maire's paper [6], where a cell centered Lagrangian scheme was developed for compressible flows. In their approach, the node velocity and numerical fluxes through cell interfaces are not independent, but constructed in a coherent manner.

### 3. Mesh velocity determination and numerical experiments

In Section 2, we have obtained the numerical fluxes. What remained is the construction of mesh velocity. In fact, we have obtained the mesh velocity on a material interface which is just the velocity of fluid by the Lagrangian method. The mesh velocity within each single component flow need to be determined as that in [7].

Example 1: Bubble/shock interaction

We consider the test example of Karni and Quirk, which has also been tested by Abgrall using the quasiconservative approach. This test example could not be simulated correctly by a number of existing methods because its complexity of the wave structure. The results are shown in Fig 1-Fig 3.
Example 2: Richtmyer-Meshkov instability (see, e.g., [7])

We consider a shock tube with length 6 and height 1 in a two-dimensional domain. The initial domain has a uniform 480-80 mesh points. In this problem, the initial condition is composed of an interface separating two fluids with different densities and a shock wave impending the interface, the initial setup is as in [7]. It is well-known that this interface become instable after the passage of the shock wave. The results are shown in Fig. 4 and Fig. 5 for the mesh distributions and density at time t=9.0, a clear interface is obviously captured.

### 4 Conclusion

Based on the integral form of fluid dynamic equations, an adaptive moving mesh BGK scheme has been developed for multi-material flow computations. To keep the sharpness of the material interface, the Lagrangian mesh velocity and an approximate Riemann solution is used for the flux evaluation. The numerical solutions demonstrate the accuracy and efficiency of the scheme for multi-material flow computation.

### Acknowledgments

This work was supported by the Special Funds for Major State Basic Research Projects (Grant No. 2005CB321700), CAEP (Grant No.2009B0202021), and NSFC (Grant No. 1097-1016).

### References