A Study of Similarity Solutions for the Unsteady Natural Convection Flow above a Semi-Infinite Heated Horizontal Porous Plate with Transpiration

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Abstract In the study of similarity solutions of unsteady laminar natural convection boundary layer flow above a heated horizontal semi-infinite porous plate, four different similarity cases arise and here we will present one of them. The governing boundary layer equations are simplified first by using the Boussinesq approximation. Secondly, similarity techniques are adopted on the basis of detailed analysis in order to reduce the coupled partial differential equations into a set of ordinary differential equations. Under the considered condition we have investigated the effect of suction and blowing on the flow and temperature fields and on other flow factors like skin friction and heat transfer coefficients. Sixth order R-K method is used to solved the simplified equations and the obtained numerical results are displayed graphically for some selected values of the controlling parameters provided by the similarity transformation. It is observed that a small value of suction or blowing play an important role on the patterns of flow and temperature fields as well as on the pressure distribution, skin friction and heat transfer coefficients.

Keywords Similarity solutions, Natural convection, Boussinesq approximation, Porous plate, Transpiration.

1. Introduction

When fluid flow is only due to the density differences caused from the temperature gradients without the influence of any external force, the flow is termed as natural or free convection flow. Such a flow is generated due to the buoyancy effects which is observed in many heat transfer processes in nature and is applied in many technological applications. The theoretical, experimental and numerical analysis for the natural convection boundary layer flow above the upper surface of isothermal or heated horizontal flat plates have been carried out widely by many authors (viz. [1], [2], [3], [4], [5], [6]). In such a boundary layer flow, the buoyancy force normal to the surface is balanced by a spatially varying pressure field, whose gradient parallel to the surface being such as to cause the boundary layer motion. The boundary layer grows from each edge of the plate and that the boundary layer motion being normal to the corresponding edge. Collisions between opposing boundary layer flows occur above the plate surface. Subsequent to collision, the fluid contained in the boundary layer forms a rising buoyant plume. In fact, boundary layer analyses do not predict any flow separation on the plate surface but practically the problem of boundary layer control has become very important factor for flow separation, owing to reduce pressure drag and to attain high lift. In actual application, the transpiration process is often necessary to prevent such a separation of the boundary layer. The effect of transpiration (suction or blowing) on free convection flow consists in the removal of decelerated particles from the boundary layer before they are given a chance to cause separation. It is also well established that the suction or blowing of fluid through a horizontal surface can significantly modify the flow field and affect the heat transfer rate for forced as well as free convection. A wide range of investigations for the effect of blowing and suction on free convection flow and heat transfer about horizontal plate were carried out extensively by [7], [8], [9] and many others.

Further, a vast literature of similarity solution has appeared in the area of fluid mechanics, heat transfer, and mass transfer, etc. as it is one of the important means for the reduction of a number of independent variables with simplifying assumptions. It is revealed that the similarity solution, which being attained for some suitable values of different parameters, might be thought of being the solution of the natural convection boundary-layer context either near the leading edge or far away in the downstream. Using a ‘parameter concerned pseudo-similarity technique’ of time and coordinates, Cheng and Huang [10] studied the unsteady laminar boundary layer flow and heat transfer in the presence and absence of heat source or sink on a continuous moving and stretching isothermal surface with suction and blowing. An analysis is performed by Aydin and Kayato [11] for the laminar boundary layer flow over a porous horizontal flat plate, particularly, to study the effect of uniform suction/injection on the heat transfer. Using the constant surface temperature thermal boundary condition they also investigated the effect of Prandtl number on heat transfer.

Recently, Hossain and Mojumder [12] presented the similarity solution for the steady, laminar, free convection boundary layer flow generated above a heated horizontal rectangular surface. They investigated the effect of suction and blowing on fluid flow and heat transfer as well as skin friction coefficients.

Further, in the study of complete similarity solutions of the unsteady laminar natural convection boundary layer flow above a heated horizontal semi-infinite porous plate (Mojumder et al. [13]), four different similarity cases arise
of which they studied one of the cases (both \( \frac{dA(\tau)}{dT} \) and \( dB(\xi) \) are finite consents).

Here we will present another of the four similarity cases. Under the considered case, we will investigate the effects of suction and blowing on the flow and temperature fields and other flow parameters like pressure distribution, skin friction and heat transfer coefficients. We will also predict how very small suction or blowing plays an important role on the effect of these parameters as well.

2. Governing equations

The simplified form of governing boundary layer equations of laminar two-dimensional unsteady flow over a semi-infinite heated horizontal porous surface in dimensionless form are as follows:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\Pr} \frac{\partial^2 u}{\partial y^2} &= \frac{\kappa}{\rho} \frac{\partial^2 u}{\partial \xi^2} \\
\frac{\partial T}{\partial \tau} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{1}{\Pr} \frac{\partial^2 T}{\partial \xi^2} \\
\end{align*}
\]

where \( g = \pm g, v_s = \frac{\mu_s}{\rho_s} \) are all constants, \( \Pr = \frac{\mu_s C_p}{k} \).

The boundary conditions to be imposed in order to determine the solution of the boundary layer equations (1) – (4) are given by

\[(i) \quad u = 0, v = v_s \text{ at } y = 0, \quad (ii) \quad T = T_w \Rightarrow \theta = 1 \text{ at } y = 0 \]
\[(iii) \quad u = 0, \text{ as } y \to \infty, \quad (iv) \quad T = T_w \Rightarrow \theta = 0 \text{ as } y \to \infty \]

3. Similarity transformations

To reduce the above system of equations into suitable forms we adopt the method of similarity solutions. Hence the following substitutions are introduced—

\[\tau = t, \xi = x, \phi = \frac{y}{\gamma(x,t)}, u = \frac{\partial \psi}{\partial \xi}, v = -\frac{\partial \psi}{\partial \tau}, \]
\[\nabla \theta = \Delta T(\tau, \xi) U(\tau, \xi) \theta(\tau, \xi, \phi), \Delta T = T_w(\tau, \xi) - T, \]
\[\psi(\tau, \xi, \phi) = \gamma(\tau, \xi) U(\tau, \xi) F(\tau, \xi, \phi), \]
\[p(\tau, \xi) = p(\tau, \xi) G(\tau, \xi, \phi), \]
\[\rho - \rho_s = -\rho_s \beta \Delta T(\tau, \xi) \theta(\tau, \xi, \phi). \]

with the traditional substitution \( \int_{\xi}^{\tau} U(\tau, \xi) d\phi = F(\tau, \xi, \phi). \)

In view of above transformations, equations (1) to (4) become

\[
\begin{align*}
u \frac{\partial F_{\phi\phi}}{\partial \phi} + a_0 \phi F_{\phi} + \frac{1}{2} (a_1 + a_2) F_{\phi}^2 - a_4 F_{\phi}^2 + a_5 G - a_6 \phi G_{\phi} &= \frac{1}{2} \left( a_1 + a_2 \right) F_{\phi}^2 \\
a_4 F_{\phi} + a_5 G &= a_1 G - a_6 \phi G_{\phi} \\

\end{align*}
\]

\[
\begin{align*}
\theta \frac{\partial \theta_{\phi}}{\partial \phi} + a_0 \phi \theta_{\phi} + \frac{1}{2} (a_1 + a_2) \theta_{\phi}^2 - (a_4 + a_5) F_{\phi}^2 + (a_2 + a_4) F_{\phi}^2 + a_0 \phi \theta_{\phi} &= 0 \\

\end{align*}
\]

where \( F(\tau, \xi, \phi), \theta(\tau, \xi, \phi) \) and \( G(\tau, \xi, \phi) \) are assumed at this stage to be function of \( \phi \) alone and

\[(i) \quad a_0 = \gamma \xi^2, \quad (ii) \quad a_1 = (\gamma^2 U) \xi, \quad (iii) \quad a_2 = \gamma^2 U \xi, \]
\[(iv) \quad a_4 = -2 \xi^{-1}, \quad (v) \quad a_5 = \frac{\gamma^2 U}{U}, \]
\[(vi) \quad a_6 = \frac{\gamma^2 \rho_s}{\rho U}, \quad (vii) \quad a_7 = \frac{\gamma \xi \rho_s}{\rho U} \]

Here \( \psi = -\frac{\partial \psi}{\partial \xi} \) represents the non-zero wall velocity called the suction or blowing velocity normal to the porous surface, so that fluid can either be sucked or blown through it. Physically, \( \psi = 0 \) and \( \psi > 0 \) represent the suction and blowing velocity through the porous surface, respectively. For uniform suction (or blowing) \( \psi = 0 \) is constant. However, \( \psi = 0 \) implies that the surface is impermeable to the fluid.

The transformed boundary conditions are now

\[F(0) = F_{\phi}(0) = 0, F_{\phi}(\infty) = 0, \theta(0) = 1, \theta(\infty) = 0 \]

The equations (9) furnish us with the conditions under which similarity solutions are obtained provided that all \( a \)'s must be constants. In view of conditions (i) – (v) in equation (9), we have \( \gamma^2 U = a_1 \xi + A(\tau) \) and \( \gamma^2 = 2a_4 \tau + B(\xi) \), where \( A(\tau) \) is either a function of \( \tau \) or constant and \( B(\xi) \) is either a function of \( \xi \) or constant.

The above two relations yield

\[
\frac{dA}{d\tau} + \frac{dB}{d\xi} = (2a_0 + a_1)(a_4 - a_2) \]

Therefore, the forms of the similarity equations, the scale factors \( U(\tau, \xi) \) and \( \gamma(\tau, \xi) \) entirely depend on the equation (11).

3.1. Similarity case

Equation (11) yields possibilities of four similarity cases, of which when both \( \frac{dA}{d\tau} \) and \( \frac{dB}{d\xi} \) are zero is considered here. Thus we have \( \gamma^2 = 2a_4 \tau + B \) and \( U = \frac{a_1 \xi + A}{2a_4 \tau + B}. \)

Substituting these in the conditions (i) – (x) of equation (9)
yields the relations between the constants as follows: \( a_0 \) and \( a_1 \) are arbitrary, \( a_2 \), \( a_3 \) and \( a_4 \) are disposable constants and \( a_2 = a_0 \), \( a_4 = -2a_0 \), \( a_5 = 0 \), \( a_6 = -2a_0 \), \( a_7 = a_1 \). Substituting the constants and choosing \( F = \alpha f \), \( \phi = \alpha \eta \) and \( G = \alpha \bar{G} \), above equations (6) – (8) reduce to \( f_{\eta \eta} + (\eta + \beta f + f_u) f_{\eta} + (2 - \beta) f_u = \bar{G} \) (12) \( \bar{G} = \theta \) (13) \( P^{-1} \theta_{\eta \eta} + (\eta + \beta f + f_u) \theta_\eta + 2(2 - \beta) f_u \theta = 0 \) (14) where it is also chosen that \( a_1 = a_2 \), \( \frac{a_2 a_4}{\nu} = 1 \), \( a_1 = \beta \), \( \frac{a_2 a_4}{\nu} = 1 \), \( \frac{\gamma \nu}{\sqrt{\alpha_2 u}} = f_w \) and for a purely free convection flow \( U = U_f = \left[ 2 \beta^2 \left( -g \beta L \right) \right]^{1/2} \) is called free convection velocity associated with the local characteristic length \( L = \frac{x}{\eta} + \frac{x}{\eta} \) or \( x + x_0 \). The transformed boundary conditions are: \( f(0) = f_\eta(0) = f_\eta(\infty) = 0 \), \( \theta(0) = 1 \), \( \theta(\infty) = 0 \); \( \bar{G}_\eta(0) = 1 \), \( \bar{G}_\eta(\infty) = 0 \) (15) where the boundary conditions for \( \bar{G} \) is obtained from that described for \( \theta \) according to equation (13). The velocity components, the skin friction and the local heat transfer coefficients related to the equations (12) – (14) are given by \( u = U_f f_\eta(\eta) \), \( v = \frac{\nu}{L} \sqrt{\frac{R_e \alpha}{2 \beta L}} \left( \beta \left( f - \eta f_u \right) + f_u \right) \), \( \tau_u = \frac{\mu U_f R_e \alpha \frac{1}{2}}{\sqrt{\beta L}} f_{\eta \eta}(0) \) and \( q_u = \frac{k \Delta T \alpha \frac{1}{2}}{\sqrt{\beta L}} \theta_\eta(0) \). 

4. Numerical Solution and Discussion

To obtain the solution of the (12) – (14) with boundary conditions (15), we have used the sixth order R-K shooting method. The trial and error process is taken come through Nachtsheim & Swigert iteration technique. The solution thus obtained in terms of the similarity variables are plotted and tabulated. The effect of suction parameter \( f_u \) and Prandtl number \( Pr \) on velocity \( f_u \), temperature \( \theta \) and pressure \( \bar{G} \) are plotted in Fig. 1 through Fig. 6. Also their effects on the skin friction and heat transfer coefficients are tabulated in Table 1. Throughout the whole numerical calculations \( \beta \) is taken to be the fixed value 0.53.

Figs. 1 and 2 represent the effect of \( f_u \) and \( Pr \) on the velocity profiles, respectively. From Fig.1 it is observed that before \( \eta \leq 2.0 \) the velocity remains negative and with the increase in the suction parameter \( f_u \), the velocity decreases in this area. Whereas, the positive velocity is observed outside this region \( (\eta > 2.0) \).

Fig. 1: Velocity profiles for different values of the suction parameter \( f_u \) with fixed value of \( Pr = 0.71 \).

From Fig. 2 we see that the negative velocity is observed in the region where \( \eta \leq 2.3 \) and on its right the velocity become positive. With the increase in \( Pr \), the velocity decreases i.e., in the negative velocity region, for small \( Pr \) the velocity is higher than those of large values of \( Pr \). In the positive velocity region, the velocities are small in comparison to the negative velocities.

Fig. 2: Velocity profiles for different values of the Prandtl number \( Pr \) with fixed value of \( f_u = 0.3 \).

Figs. 3 and 4 represent the effect of \( f_u \) and \( Pr \) on the temperature profiles respectively. From Fig. 3 we see that in the region where \( \eta < 1.25 \), the temperature decreases with the increasing \( \eta \). Again, it reverses the direction and finally the temperature asymptotically become zero for large value of \( \eta \). Here it is observed that temperature decreases with the increase in the suction parameter \( f_u \).

Fig. 3: Temperature profiles for different values of the suction parameter \( f_u \) with fixed value of \( Pr = 0.71 \).

From Fig. 4 it is observed that in the region where \( \eta < 1.3 \), the temperature decreases with the increase in \( Pr \) except when \( Pr \) is 7.0. After that temperature increases with the increase in \( Pr \), it again decreases when \( Pr \) increases. This peculiar behavior for \( Pr = 7.0 \) is observed may due to the constituents of the fluid.

Fig. 4: Temperature profiles for different values of the Prandtl number \( Pr \) with fixed value of \( f_u = 0.3 \).
Figs. 5 and 6 representing the effects of \( f_w \) and Pr respectively on the pressure variable \( \overline{g} \). The initial value of the dependent variable \( \overline{g} \) is not known rather it was subject to guessing to satisfy the boundary conditions at the other end. As a result the graphs for \( \eta = 0 \) (on the wall) starts from different points and for small values of \( \eta \) (close to the wall) the patterns are different from the other values of \( \eta \) as shown in Figs. 5 and 6. As we observe, pressure decreases with the increase of both \( f_w \) and Pr when \( \eta \) lies between 0.5 to 2.5 and become negative. Afterwards, it increases and become asymptotically zero for large value of \( \eta \).

![Fig. 5: Pressure distributions for different values of the suction parameter \( f_w \) with fixed value of Pr = 0.71.](image)

![Fig. 6: Pressure distributions for different values of the Prandtl number Pr with fixed value of \( f_w = 0.3 \).](image)

The values proportional to the coefficients of skin friction and heat transfer are tabulated in Table 1. From this table it is seen that for fixed Pr with increase in the \( f_w \), both the coefficients of skin friction and heat transfer decrease, whereas, the coefficient of skin friction increases and the coefficient of heat transfer decreases for fixed value of \( f_w \) with increase in Pr. Unfortunately, no experimental data is available to us to verify our numerical results.

Table 1: Variation of the coefficients of skin friction and heat transfer with \( f_w \) and Pr

<table>
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<tr>
<th>( f_w )</th>
<th>Pr</th>
<th>( f_{\eta \eta} (0) )</th>
<th>( \theta_p (0) )</th>
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5. Conclusions

Four different similarity cases arise with the choice of \( \frac{dA}{d\tau} \) and \( \frac{dB}{d\zeta} \) either zero or constant. Similarity solution for one case is being studied in this paper. Further study is necessary to solve rest of the cases.

References